Technical Comments

Comment on "A New Integral Calculation of Skin Friction on a Porous Plate"

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In a recent paper[1] it is claimed that a refined Kármán Pohlhausen (K.P.) method, based on a double integration, in the direction normal to the external mainstream, of the incompressible, zero pressure gradient boundary-layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

leads to accurate and reliable results for the problem of determining the skin friction on a porous plate. The purpose of this Comment is to indicate that the method used is not new, but that it is a standard variation of the K.P. method and indeed is a variation which has already been explored.[2]

From the incompressible boundary-layer equation an ordinary differential equation for the boundary layer thickness $\delta(x)$ may be obtained by multiplying by $y^m u^n$, where m and n are integers, and integrating the resulting equation with respect to y from 0 to δ. By substituting a velocity profile which is defined in terms of δ an ordinary differential equation with δ as dependent variable can be obtained. If the assumed velocity profile contains k unknown functions then k equations can be obtained by taking k possible combinations of m and n. The combination $m = n = 0$ has special physical significance and the K.P. ordinary differential equation can in fact be derived without reference to the boundary-layer equation itself.[3] A comprehensive account of the development of the subject may be found in Ref. 2; the concensus of opinion, evolved with experience over the years, is that such methods will in general prove reliable for flows in which the pressure gradient is favorable but will be unreliable for flows with adverse pressure gradient. More sophisticated integral techniques do not suffer from this disadvantage.[4]

If Eq. (1) be multiplied by y, if the resulting equation be integrated over the boundary layer, and if v be eliminated by means of the equation

$$v(x, y) = v_w(x) - \int_0^y \frac{\partial u}{\partial x}\, dy$$

then we obtain, after simple integration by parts, the equation

$$\int_0^\delta y\frac{\partial}{\partial x}u^2 dy + \int_0^\delta dy\, u(x, y)\int_0^y \frac{\partial}{\partial x}u(x, \theta)\, d\theta = \delta u_0 \int_0^\delta \frac{\partial u}{\partial x}dy - u_0\nu - v_w\left[\delta u_0 - \int_0^\delta u dy\right]$$

This is the equation to which Eq. (2) of Ref. reduces if the first term of that equation is integrated by parts.

To see why the author's results are superior to the standard K.P. results we need only look at the form of Eq. (1) at the wall, i.e.

$$v_w \left.\frac{\partial u}{\partial y}\right|_{y=0} = \nu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \tag{2}$$

Here, a significant error is introduced into the K.P. solution since Eq. (2) is not satisfied by either of the profiles for u/u_0 selected in Ref. 1. If, on the other hand, the boundary-layer equation be multiplied by y prior to integration over the boundary layer, the error introduced near the wall will be diminished since the equation is now satisfied at the wall.

Table 1, compares, for the case of zero mass transfer, the results for τ_0, the wall shear stress, obtained with various profiles $f(\eta)$ and for various combinations of m and n.

Table 1 Comparison of skin-friction results for various K.P. formulations (case of zero mass transfer)

m	n	$f(\eta)$	$\frac{\tau_0}{\rho u_0^2}\left(\frac{u_0 x}{\nu}\right)^{1/2}$	Ref.
0	0	η	0.289	3
0	1	η	0.250[a]	1
0	0	$2\eta - 2\eta^3 + \eta^4$	0.343	3
0	1	$2\eta - 2\eta^3 + \eta^4$	0.350[a]	1
0, 1	0, 0	$1 - (1 - \eta)^3(1 + \lambda\eta)$	0.3314	5
		Exact solution	0.3321	

[a] As calculated from $\tau_0 = (\mu/\delta)u_0 f'(0)$.

References

[1] Zien, T. F., "A New Integral Calculation of Skin Friction on a Porous Plate," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1423–1425.

[2] Rosenhead, L. R., ed., *Laminar Boundary Layers*, Oxford University Press, Oxford, England, 1963, pp. 317–318 and Chapt. VI.

[3] Goldstein, S., ed., *Modern Developments in Fluid Dynamics*, Vol. 1, Dover, New York, 1965, pp. 131–132 and 157.

[4] MacDonald, D. A., "Solution of the Incompressible Boundary Layer Equations via the Galerkin Kantorovich Technique," *J. Inst. Maths. Applics.*, Vol. 6, 1970, pp. 115–130.

[5] Sutton, W. G. L., "An approximate Solution of the Boundary Layer Equations for a Flat Plate," *The Philosophical Magazine*, Vol. 23, Ser. 7, pp. 1146–1152.

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Reply by Author to D. A. MacDonald

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The statement by MacDonald that the method used in Ref. 1 is not new and it is one of the previously explored variations of the K-P method is both inaccurate and misleading. The earlier brief reply to Granville by the author[2] and a subsequent paper[3] by the author together with a careful digest of Ref. 1 actually suffice to clarify the issue. However, for the benefit of other casual readers, it is felt that the following remarks are perhaps worthwhile.

First of all, this author apparently has failed to make clear the central idea of the present method and has thus regrettably

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